

CHAOTIC DYNAMICS OF THE HEAT SEMIGROUP ON THE DAMEK-RICCI SPACES

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ABSTRACT. The Damek-Ricci spaces are solvable Lie groups and noncompact harmonic manifolds. The rank one Riemannian symmetric spaces of noncompact type sits inside it as a thin subclass. In this note we establish that for any Damek-Ricci space S , the heat semigroup generated by certain perturbation of the Laplace-Beltrami operator is *chaotic* on the Lorentz spaces $L^{p,q}(S)$, $2 < p < \infty, 1 \leq q < \infty$ and subspace-chaotic on the weak L^p -spaces. We show that both the amount of perturbation and the range of p are sharp. This generalizes a result in [18] which proves that under identical conditions, the heat semigroup mentioned above is *subspace-chaotic* on the L^p -spaces of the symmetric spaces.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

This article is inspired by a recent paper of Ji and Web ([18]) in which the authors considered the heat semigroup $T_t, t \geq 0$, generated by certain perturbation (which depends on p) of the Laplace-Beltrami operator of a Riemannian symmetric space $X = G/K$ of noncompact type. The authors in [18] have shown that T_t is chaotic on the subspace of K -invariant functions of $L^p(G/K)$ and hence is *subspace-chaotic* on $L^p(G/K)$ for $2 < p < \infty$. This poses a few clear questions: (1) Is T_t *chaotic* on the full space $L^p(G/K)$ with p in the same range? (2) Is the amount of perturbation required sharp? (3) Exactly when or for which function space the chaoticity *slips* into the subspace-chaoticity? On the other hand, a study of [18] reveals that the parabolic shape of the spectrum of the Laplacian is crucial for this non-Euclidean phenomenon, while the p -dependence of the position of the parabolic region justifies the p -dependence of the perturbation. This throws some vindication that it might be possible to extend these results to the non-symmetric generalization of the rank one Riemannian symmetric spaces of noncompact type, namely the Damek-Ricci spaces (which are also known as Harmonic NA or AN groups). We shall address these questions. (See the statements below.) We need some preparation before stating the results.

The Damek-Ricci (DR) spaces are solvable Lie groups as well as harmonic manifolds, but very rarely they are symmetric spaces. Indeed a general DR space appears as a counter example to the Lichnerowicz conjecture (see [9, 10]), citing that there are noncompact harmonic manifolds which are not symmetric spaces. However the rank one Riemannian symmetric spaces of noncompact type form a very thin subclass in the set of DR spaces (see [2]). It is well known that such a symmetric space X is realized as a quotient space G/K where G is a connected noncompact semisimple Lie group with finite centre and K is a maximal compact subgroup of G . Thus G (as well as K) has natural left action on X and functions on X can be realized as right K -invariant functions of G . One can thus use the full semisimple machinery and in particular the method of decomposing a function in K -types to tackle the questions on the function spaces of X . The lack of rotation group in a general DR space is an important difference, which offers fresh difficulties. We note in this context that the concept of radially in a general DR

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space is not connected with the group action, which is in contrast with that of the symmetric spaces where a radial function is simply a K -invariant function.

We need the following definitions to proceed: A strongly continuous semigroup on a Banach space B is a map T from $[0, \infty)$ to the space of all bounded linear operators from B to B , such that $T(0) = T_0 = I$, the identity operator on B ; for all $t, s \geq 0$, $T_{t+s} = T_t T_s$ and for all $x_0 \in B$, $\|T_t x_0 - x_0\| \rightarrow 0$, as $t \rightarrow 0$. The infinitesimal generator A of a strongly continuous semigroup T is defined by $Ax = \lim_{t \downarrow 0} \frac{1}{t} (T_t - I)x$, whenever the limit exists and we write $T_t = e^{tA}$.

- (i) A semigroup of operators $T_t, t \geq 0$ on a Banach space B is *hypercyclic* if there exists a $v \in B$ such that $\{T_t v \mid t \geq 0\}$ is dense in B .
- (ii) A point $v \in B$ is *periodic* for T_t , if there exists a $t > 0$ such that $T_t v = v$.
- (iii) The semigroup T_t is *chaotic* if it is hypercyclic and if its periodic points make a dense set in B .
- (iv) The semigroup T_t is *subspace-chaotic* if there is a closed T_t -invariant subspace $V \neq \{0\}$ of B such that $T_t|_V$ is chaotic on V .

We have followed [18] where chaos is defined in the sense of Devaney (see [13]). For a comprehensive exposition we refer to [18]. Henceforth S will denote a Damek-Ricci (DR) space. When a DR space is a rank one Riemannian symmetric space, we shall denote it by G/K . Let $-\Delta$ be the Laplace-Beltrami operator on S . Throughout this article we shall use the notation c_p for $4\rho^2/pp'$ where $p \geq 1$, $p' = p/(p-1)$ and $\rho = Q/2$, Q being the homogenous dimension of S (see section 3). If the DR space is a symmetric space then ρ coincides with the half-sum of positive roots, considered as a scalar. We shall assume that $c_\infty = 0$.

Purpose of this note is to establish that for any DR space S , the heat semigroup $T_t = e^{-(\Delta-c)t}$ with $c > c_p$, is *chaotic* on the Lorentz spaces $L^{p,q}(S)$, $2 < p < \infty, 1 \leq q < \infty$ and subspace-chaotic on the weak L^p -spaces when $2 < p \leq \infty$. We show that both the range of p and the amount of perturbation c are sharp. This generalizes one of the main results in [18] which proves that T_t under identical condition is *subspace-chaotic* on the L^p -spaces of the symmetric spaces. We recall that the Lorentz spaces are finer subdivisions of the Lebesgue spaces. Apart from the chaoticity to non-chaoticity, the use of Lorentz spaces locates another point of degeneracy, where chaoticity changes to subspace-chaoticity. This conforms the paradigm that the subspace-chaoticity is more stable than chaoticity, as the results assert that T_t is at least subspace-chaotic on $L^{p,q}(S)$ for any $1 \leq q \leq \infty$ if and only if $p > 2$ and $c > c_p$. Our main results are the following. (See section 2 for any unexplained notation.)

Theorem A. For $t \geq 0$ and $c \in \mathbb{R}$, let $T_t = e^{-t(\Delta-c)}$. Then,

- (i) for $2 < p < \infty, 1 \leq q < \infty$, T_t is chaotic on $L^{p,q}(S)$ if and only if $c > c_p$;
- (ii) for $2 < p \leq \infty$, T_t is not chaotic on $L^{p,\infty}(S)$ for any $c \in \mathbb{R}$, but subspace-chaotic if and only if $c > c_p$;
- (iii) for $2 < p < \infty, 1 \leq q \leq \infty$, T_t is not hypercyclic and not subspace-chaotic on $L^{p,q}(S)$ and on $L^\infty(S)$ if $c \leq c_p$.

Note that (ii) includes the space $L^\infty(S) = L^{\infty,\infty}(S)$. Part (iii) emphasizes the drastic changes caused by the amount of perturbation c . The next theorem establishes the sharpness of the condition $p > 2$.

Theorem B. For $t \geq 0$ and $c \in \mathbb{R}$, let $T_t = e^{-t(\Delta-c)}$.

- (i) For $1 < q \leq \infty$, T_t is not chaotic or subspace-chaotic on $L^{2,q}(S)$. If $c \leq \rho^2$ then T_t is not hypercyclic on $L^{2,q}(S)$.

- (ii) *The semigroup T_t is not hypercyclic (hence not chaotic) and not subspace-chaotic on the spaces $L^1(S)$, $L^{2,1}(S)$ and $L^{p,q}(S)$, with $1 < p < 2, 1 \leq q \leq \infty$.*

Proving the theorems only for the rank one symmetric spaces (which we recall, form a very small subclass of all DR spaces) would be somewhat simpler as there one can use the compact boundary of the space. Here instead our argument is based on the noncompact boundary. The paper is organized as follows. The general preliminaries, definitions and results related to chaos are given section 2. In section 3 we give the basic introduction to DR spaces and arrange an array of tools required for the proofs. In section 4 we prove the results stated above. Finally in section 5 we discuss existence of periodic points of the operator T_t in various function spaces.

2. NOTATION AND PRELIMINARIES

2.1. Generalities. For any $p \in [1, \infty)$, let $p' = p/(p-1)$ and $\gamma_p = (2/p-1)$. The letters \mathbb{R} , \mathbb{Q} and \mathbb{C} denote respectively the set of real numbers, rational numbers and complex numbers. For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote respectively the real and imaginary parts of z . For a set A in a measure space, $|A|$ denotes the measure of A and for a set S in a topological space, S° denotes its interior. The letters C, C_1, C_2 etc. will be used for positive constants, whose value may change from one line to another. Occasionally the constants will be suffixed to show their dependencies on important parameters.

2.2. Lorentz spaces. We shall briefly introduce the Lorentz spaces (see [16, 24, 23] for details). Let (M, m) be a σ -finite nonatomic measure space, $f : M \rightarrow \mathbb{C}$ be a measurable function and $p \in [1, \infty)$, $q \in [1, \infty]$. We define

$$\|f\|_{p,q}^* = \begin{cases} (q \int_0^\infty (td_f(t)^{1/p})^q \frac{dt}{t})^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} td_f(t)^{1/p} & \text{if } q = \infty, \end{cases}$$

where for $\alpha > 0$, $d_f(\alpha) = |\{x \mid f(x) > \alpha\}|$ is the distribution function of f . We take $L^{p,q}(M)$ to be the set of all measurable $f : M \rightarrow \mathbb{C}$ such that $\|f\|_{p,q}^* < \infty$. For $1 \leq p \leq \infty$, $L^{p,p}(M) = L^p(M)$ and $\|\cdot\|_{p,p}^* = \|\cdot\|_p$. We note that the Lorentz “norm” $\|\cdot\|_{p,q}^*$ is actually a quasi-norm which makes the space $L^{p,q}(M)$ a quasi Banach space (see [16, p. 50]). However for $1 < p \leq \infty$, there is an equivalent norm $\|\cdot\|_{p,q}$ through which it is a Banach space (see [24, Theorems 3.21, 3.22]). We shall slur over this difference, use the notation $\|\cdot\|_{p,q}$ and consider $L^{p,q}(S)$ a Banach space with this norm whenever $p > 1$ and we shall not deal with $L^{1,q}$ spaces where q is other than 1. The spaces $L^{p,\infty}(M)$ are known as the weak L^p -spaces. Thus weak L^∞ space is same as the L^∞ space. Following properties of the Lorentz spaces will be required (see [16]). Henceforth for a Banach space B , its dual space will be denoted by B^* .

- (i) Simple functions are dense in $L^{p,q}(M)$, $1 < p < \infty$, $1 \leq q < \infty$, but not in $L^{p,\infty}(M)$, $L^\infty(M)$.
- (ii) Unlike $L^{p,q}(M)$ with $q < \infty$, $L^{p,\infty}(M)$ and $L^\infty(M)$ are not separable.
- (iii) If $q_1 \leq q_2 \leq \infty$, then $L^{p,q_1}(M) \subset L^{p,q_2}(S)$ and $\|f\|_{p,q_2} \leq \|f\|_{p,q_1}$. If $q_2 < \infty$ then $L^{p,q_1}(M)$ is a dense subspace of $L^{p,q_2}(S)$.
- (iv) For $1 < p, q < \infty$, $(L^{p,q}(M))^* = L^{p',q'}(M)$; $(L^{p,1}(M))^* = L^{p',\infty}(M)$; $(L^{p,\infty}(M))^* = L^{p',1}(M) \oplus S$ where elements of S are singular functionals (see [8]) and $(L^\infty(M))^* = L^{p,1}(S) \oplus M$ where M consists of certain finitely additive measures.

2.3. Chaos and hypercyclicity. In section 1 we have defined chaos and hypercyclicity. For a detailed account we refer to [18]. (See also the references therein, in particular [4, 12, 11]). Here we shall limit ourselves to what is needed to make the article self-contained. Let B be a Banach space and B^* be its dual space. For a linear operator A on B , let $\sigma_{pt}(A)$ be its point spectrum. For a strongly continuous semigroup of operators (see section 1) $T_t, t \geq 0$ acting on B , let:

- (a) $B_0 = \{x \in B \mid \lim_{t \rightarrow \infty} T_t x = 0\}$;
- (b) B_∞ be the set of $x \in B$ such that for each $\varepsilon > 0$ there exists $w \in B$ and $t > 0$ with $\|w\| < \varepsilon$ and $\|T_t w - x\| < \varepsilon$;
- (c) B_{Per} be the set of all periodic points in B .

Following is a key result proved in [12].

Theorem 2.3.1. *Let T_t denote a strongly continuous semigroup of operators on a separable Banach space B . If B_∞ and B_0 are dense in B , then T_t is hypercyclic.*

We also have the following necessary conditions (see [12, 11]) for T_t being hypercyclic/chaotic on B :

Proposition 2.3.2. *Let $T_t, t \geq 0$ be a semigroup of operators generated by A in a Banach space B .*

- (i) *If T_t is chaotic on B then the intersection of the point spectrum of A with $i\mathbb{R}$ is infinite.*
- (ii) *If T_t is hypercyclic on B then for the adjoint operator A^* of A on the dual space B^* , $\sigma_{pt}(A^*) = \emptyset$.*
- (iii) *If T_t is hypercyclic on B then for any $\phi \in B^*$, $\phi \neq 0$ the orbit $\{T_t^* \phi \mid t \geq 0\}$ is unbounded.*

Remark 2.3.3. It is clear that these conditions above can actually detect when T_t is not even subspace-chaotic on a Banach space B . Precisely, if $\sigma_{pt}(A) \cap i\mathbb{R}$ is finite or $\sigma_{pt}(A^*) \neq \emptyset$ or there exists nonzero $\phi \in B^*$ such that $\{T_t^* \phi \mid t \geq 0\}$ is bounded then T_t is not only non-chaotic, it is non-subspace-chaotic on B .

3. DAMEK-RICCI SPACES

To make the article self-contained we shall briefly introduce the DR spaces in this section. Details can be retrieved from [3, 2, 23, 19]. Along the way, we shall also prepare all technical tools required to prove the main theorems. While most of these are known to the experts, it may not be available in this form. In particular Lemma 3.2.1 is new.

3.1. DR Spaces. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a H -type Lie algebra where \mathfrak{v} and \mathfrak{z} are vector spaces over \mathbb{R} of dimensions m and l respectively. Indeed \mathfrak{z} is the centre of \mathfrak{n} and \mathfrak{v} is its ortho-complement with respect to the inner product of \mathfrak{n} . Then we know that m is even. The group law of $N = \exp \mathfrak{n}$ is given by

$$(X, Y).(X', Y') = ((X + X', Y + Y' + \frac{1}{2}[X, X']) \quad X \in \mathfrak{v}, Y \in \mathfrak{z}.$$

We shall identify \mathfrak{v} , \mathfrak{z} and N with \mathbb{R}^m , \mathbb{R}^l and $\mathbb{R}^m \times \mathbb{R}^l$ respectively. The group $A = \{a_t = e^t \mid t \in \mathbb{R}\}$ acts on N by nonisotropic dilation: $\delta_t(X, Y) = (e^{t/2}X, e^tY)$. Let $S = NA = \{(X, Y, a_t) \mid (X, Y) \in N, t \in \mathbb{R}\}$ be the semidirect product of N and A under the action above. The group law of S becomes:

$$(X, Y, a_t)(X', Y', a_s) = (X + a_{t/2}X', Y + a_tY' + \frac{a_{t/2}}{2}[X, X'], a_{t+s}).$$

It then follows that $\delta_t(X, Y) = a_t n a_{-t}$, where $n = (X, Y)$. The Lie group S is solvable, connected and simply connected with Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ and is nonunimodular. The homogenous dimension of S is $Q = m/2 + l$. For convenience we shall also use the notation $\rho = Q/2$. We note that ρ corresponds

to the half-sum of positive roots when $S = G/K$, a rank one symmetric space of noncompact type. The group S is equipped with the left-invariant Riemannian metric d induced by

$$\langle (X, Z, \ell), (X', Z', \ell') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + \ell \ell'$$

on \mathfrak{s} . The associated left invariant Haar measure dx on S is given by

$$(3.1.1) \quad \int_S f(x) dx = \int_{N \times A} f(na_t) e^{-Q_t} dt dn,$$

where $dn(X, Y) = dX dY$ and dX, dY, dt are Lebesgue measures on \mathfrak{v} , \mathfrak{z} and \mathbb{R} respectively. For an element $x = na_t \in S$, we shall use the notation $A(x) = t$. We denote the Laplace-Beltrami operator associated to this Riemannian structure by $-\Delta$.

The group S can also be realized as the unit ball

$$B(\mathfrak{s}) = \{(X, Z, \ell) \in \mathfrak{s} \mid |X|^2 + |Z|^2 + \ell^2 < 1\}$$

via a Cayley transform $\mathcal{C} : S \longrightarrow B(\mathfrak{s})$ (see [2, p. 646–647] for details). For an element $x \in S$, let

$$|x| = d(\mathcal{C}(x), 0) = d(x, e) = \log \frac{1 + \|\mathcal{C}(x)\|}{1 - \|\mathcal{C}(x)\|},$$

where e is the identity element of S . In particular $d(a_t, e) = |t|$.

A function f on S is called *radial* if for all $x, y \in S$, $f(x) = f(y)$ if $d(x, e) = d(y, e)$. For a function space $\mathcal{L}(S)$ on S we denote its subspace of radial functions by $\mathcal{L}(S)^\#$. For a suitable function f on S its radialization $\mathcal{R}f$ is defined as

$$(3.1.2) \quad \mathcal{R}f(x) = \int_{S_\nu} f(y) d\sigma_\nu(y),$$

where $\nu = |x|$ and $d\sigma_\nu$ is the surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S \mid d(y, e) = \nu\}$ normalized by $\int_{S_\nu} d\sigma_\nu(y) = 1$. It is clear that $\mathcal{R}f$ is a radial function and if f is radial then $\mathcal{R}f = f$. We recall the following properties of the operator \mathcal{R} (see [9, 3]):

- (1) $\langle \mathcal{R}\phi, \psi \rangle = \langle \phi, \mathcal{R}\psi \rangle = \langle \mathcal{R}\phi, \mathcal{R}\psi \rangle$ for all $\phi, \psi \in C_c^\infty(S)$;
- (2) $\mathcal{R}(\Delta f) = \Delta(\mathcal{R}f)$.

Since $|\mathcal{R}f| \leq \mathcal{R}|f|$ and by (1) above, $\int_S f(x) dx = \int_S \mathcal{R}f(x) dx$, we have $\|\mathcal{R}f\|_1 \leq \|f\|_1$. Interpolating ([24, p. 197]) with the trivial L^∞ -boundedness of \mathcal{R} we have,

$$\|\mathcal{R}f\|_{p,q} \leq \|f\|_{p,q}, 1 < p < \infty, 1 \leq q \leq \infty.$$

For two measurable functions f and g on S we define their convolution as (see [15, p. 51]):

$$f * g(x) = \int_S f(y) g(y^{-1}x) dy = \int_S f(xy^{-1}) g(y) e^{QA(y)} dy$$

where $e^{-QA(\cdot)}$ is the modular function of S . For a measurable function g on S let $g^*(x) = g(x^{-1})$. If g is radial then $g^* = g$ as $d(x, e) = d(x^{-1}, e)$. It is easy to see that for measurable functions f, g, h on S , $\langle f * g, h \rangle = \langle f, h * g^* \rangle$ if both sides make sense.

The Poisson kernel $\mathcal{P} : S \times N \longrightarrow \mathbb{R}$ is defined by $\mathcal{P}(na_t, n_1) = \wp_{a_t}(n_1^{-1}n)$ where

$$(3.1.3) \quad \wp_{a_t}(n) = \wp_{a_t}(V, Z) = Ca_t^Q \left(\left(a_t + \frac{|V|^2}{4} \right)^2 + |Z|^2 \right)^{-Q}, \quad n = (V, Z) \in N.$$

In particular

$$(3.1.4) \quad \wp_1(n) = \wp_{a_0}(n) = C[(1 + \frac{|V|^2}{4})^2 + |Z|^2]^{-Q}.$$

The value of C is adjusted so that $\int_N \wp_1(n)dn = 1$ (see [3, (2.6)]). We note that $\wp_{a_t}(n) = \wp_{a_t}(n^{-1}) = \wp_1(a_{-t}na_t)e^{-Q_t} = \wp_1(\delta_{-t}(n))e^{-Q_t}$. The complex power of Poisson kernel \mathcal{P}_λ is defined by

$$\mathcal{P}_\lambda(x, n) = \mathcal{P}(x, n)^{\frac{1}{2} - \frac{i\lambda}{Q}}.$$

Then for each fixed $n \in N$, $\Delta \mathcal{P}_\lambda(x, n) = (\lambda^2 + \rho^2)\mathcal{P}_\lambda(x, n)$. The Poisson transform of a function F on N is defined as (see [3])

$$\mathfrak{P}_\lambda F(x) = \int_N F(n)\mathcal{P}_\lambda(x, n)dn.$$

It follows that $\Delta \mathfrak{P}_\lambda F = (\lambda^2 + \rho^2)\mathfrak{P}_\lambda F$. For $\lambda \in \mathbb{C}$, the elementary spherical function ϕ_λ is given by

$$\phi_\lambda(x) = \int_N \mathcal{P}_\lambda(x, n)\mathcal{P}_{-\lambda}(e, n)dn = \int_N \mathcal{P}_{-\lambda}(x, n)\mathcal{P}_\lambda(e, n)dn.$$

and we have (see [3, Prop 4.2]),

$$(3.1.5) \quad \phi_\lambda(x^{-1}y) = \int_N \mathcal{P}_\lambda(x, n)\mathcal{P}_{-\lambda}(y, n)dn.$$

It follows that ϕ_λ is a radial eigenfunction of Δ with eigenvalue $(\lambda^2 + \rho^2)$ satisfying $\phi_\lambda(x) = \phi_{-\lambda}(x)$, $\phi_\lambda(x) = \phi_\lambda(x^{-1})$, $\phi_{-i\rho} \equiv 1$ and $\phi_\lambda(e) = 1$. We have the following asymptotic estimate of ϕ_λ (see [2]):

$$(3.1.6) \quad |\phi_{\alpha+i\gamma_p\rho}(x)| \asymp e^{-(2\rho/p')|x|}, \quad \alpha \in \mathbb{R}, 0 < p < 2.$$

The estimate above degenerates when $p = 2$, i.e. when $\gamma_p = 0$ and in this case we have $\phi_0(x) \asymp (1 + |x|)e^{-\rho|x|}$. If $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $t \geq 1$ then the Harish-Chandra series for ϕ_λ implies,

$$(3.1.7) \quad \phi_\lambda(a_t) = e^{-\rho t}[c(\lambda)e^{i\lambda t} + c(-\lambda)e^{-i\lambda t} + E(\lambda, t)], \text{ where } |E(\lambda, t)| \leq C_\lambda e^{-2t},$$

where $c(\lambda)$ is the Harish-Chandra c -function. See [17, (3.11)] for a proof of the above for the symmetric spaces. The proof works *mutatis mutandis* for general Damek-Ricci spaces. Let $C_0(S)$ be the space of continuous functions vanishing at infinity with supremum norm. Then $C_0(S)$ is a separable Banach space and a non-dense subspace of $L^\infty(S)$. For $p \geq 1$, let

$$S_p = S_{p'} = \{z \in \mathbb{C} \mid |\Im z| \leq |\gamma_p|\rho\}.$$

Let S_p° and ∂S_p respectively be the interior and the boundary of the strip S_p . We recall that (see [23, 19, 20]):

- (a) $\phi_\lambda \in L^\infty(S)$ if and only if $\lambda \in S_1$;
- (b) $\phi_\lambda \in C_0(S)$ if and only if $\lambda \in S_1^\circ$;
- (c) for $1 < p < 2$, $1 \leq q < \infty$, $\phi_\lambda \in L^{p',q}(S)$ if and only if $\lambda \in S_p^\circ$;
- (d) for $1 < p < 2$, $\phi_\lambda \in L^{p',\infty}(S)$ if and only if $\lambda \in S_p$;
- (e) for $\lambda \in S_2 = \mathbb{R}$, $\phi_\lambda \in L^{2,\infty}(S)$ if and only if $\lambda \neq 0$;
- (f) for $\lambda \in S_2 = \mathbb{R}$, $\phi_\lambda \notin L^{2,q}(S)$ for any $q < \infty$.

The spherical Fourier transform of a function f is defined by $\widehat{f}(\lambda) = \int_S f(x)\phi_\lambda(x)dx$ whenever the integral converges. The estimates of ϕ_λ given above determines the domain of the spherical Fourier transform of functions in different Lebesgue and Lorentz spaces. Indeed,

- (a) For $f \in L^1(S)$, \widehat{f} extends as an analytic function on S_1° which is continuous on its boundary; for a complex measure μ on S , $\mu(\lambda) = \int \phi_\lambda(x)d\mu(x)$ behaves the same way;
- (b) for $f \in L^{p,q}(S)$, with $1 < p < 2$ and $1 < q \leq \infty$, \widehat{f} extends as an analytic function on S_p° ;
- (c) for $f \in L^{p,1}(S)$, with $1 < p < 2$, \widehat{f} extends as an analytic function on S_p° which is continuous on its boundary;

(d) for $f \in L^{2,1}(S)$, \widehat{f} is continuous on nonzero real numbers.

For a measurable function f on S following [3] we define its (Helgason-type) Fourier transform by

$$\widetilde{f}(\lambda, n) = \int_S f(x) \mathcal{P}_\lambda(x, n) dx,$$

whenever the integral converges. If g is radial then, $\widetilde{f * g}(\lambda, n) = \widetilde{f}(\lambda, n) \widehat{g}(\lambda)$ whenever both sides make sense.

For $f \in L^{p,q}(S)$, $1 < p < 2$, $1 \leq q \leq \infty$ (respectively $f \in L^1(S)$), $\lambda \mapsto \widetilde{f}(\lambda, n)$ is a holomorphic function on S_p° (respectively on S_1°) for every fixed $n \in N_1$ where N_1 is a subset of N of full measure (see [23, Theorem 3.4, Theorem 5.4]). The argument in [23] also shows that if μ is a (bounded) complex measure on S , then $\widetilde{\mu}(\lambda, n)$ is a holomorphic function on S_1° for every fixed $n \in N_1$ for N_1 as above. The inversion formula is the following (see [23]). We recall that $|c(\lambda)|^{-2}$ is the Harish-Chandra Plancherel measure.

Proposition 3.1.1. *Let $f \in L^{p,q}(S)$, $p \in (1, 2)$, $q \geq 1$ or $f \in L^1(S) \cup L^2(S)$. If $\widetilde{f} \in L^1(N \times \mathbb{R}, |c(\lambda)|^{-2} d\lambda dn)$, then for almost every $x \in S$*

$$f(x) = C \int_{N \times \mathbb{R}} \widetilde{f}(\lambda, n) \mathcal{P}_{-\lambda}(x, n) |c(\lambda)|^{-2} d\lambda dn.$$

If f is radial then $\widetilde{f}(\lambda, n) = \widehat{f}(\lambda) \mathcal{P}_\lambda(e, n)$ and the inversion formula reduces to

$$(3.1.8) \quad f(x) = C \int_{\mathbb{R}} \widehat{f}(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda.$$

We have the following estimate of the Fourier transform vis-a-vis the Poisson transform (see [19, Theorem 1.1]) which will be used in the proof Theorem A. For $p > 2$, $p' < r < p$, $\alpha \in \mathbb{R}$ and $f \in L^{p',\infty}(S)$,

$$(3.1.9) \quad \|\widetilde{f}(\alpha + i\gamma_r \rho, \cdot)\|_{L^r(N)} \leq C \|f\|_{p',\infty}$$

which by duality is equivalent to (for any $F \in L^{r'}(N)$),

$$(3.1.10) \quad \|\mathfrak{P}_{\alpha+i\gamma_r\rho} F\|_{p,1} \leq C \|F\|_{L^{r'}(N)}.$$

We conclude this subsection defining the Harish-Chandra Schwartz spaces on S . For $1 \leq p \leq 2$, the L^p -Schwartz space $C^p(S)$ is defined (see [2, 14]) as the set of C^∞ -functions on S such that

$$\gamma_{r,D}(f) = \sup_{x \in S} |Df(x)| \phi_0^{-2/p} (1 + |x|)^r < \infty,$$

for all nonnegative integers r and left invariant differential operators D on S . We recall that (see [20]) for $1 \leq p \leq 2$, $C^p(S)$ is dense in $L^{p,1}(S)$ and hence in $L^{p,q}(S)$ for $1 \leq q < \infty$, but not in $L^{p,\infty}(S)$.

3.2. Herz's criterion. Let μ be a nonnegative radial finite measure. We consider the right convolution operator T_μ defined on the measurable functions on S by $T_\mu : f \mapsto f * \mu$, whenever it makes sense. We have the following *Herz's criterion* for the Lorentz spaces. In [2, Theorem 3.3]) (see also [6],[7, Theorem 3.2], [19, Proposition 4.1]) a more general result is obtained for the Lebesgue spaces on S .

Lemma 3.2.1. *Let $1 < p < \infty$, $1 \leq q \leq \infty$ be fixed. If a nonnegative radial measure μ satisfies $\int_S \phi_{i\gamma_p \rho}(x) d\mu(x) < \infty$, then T_μ is a bounded operator from $L^{p,q}(S)$ to itself and the operator norm of T_μ satisfies $\|T_\mu\|_{L^{p,q} \rightarrow L^{p,q}} \leq \int_S \phi_{i\gamma_p \rho}(x) d\mu(x)$.*

Proof. For $x \in S$, we define $(R_p(x)f)(y) = e^{-Q/pA(x)}f(yx)$ where for $x = na_t$, $A(x) = t$. We shall show that $\|R_p(x)f\|_{p,q} = \|f\|_{p,q}$ for any fixed $x \in S$. Indeed for any $\alpha > 0$,

$$\begin{aligned} d_{R_p(x)f}(\alpha) &= |\{y \in S \mid |(R_p(x)f)(y)| > \alpha\}| \\ &= |\{y \in S \mid |f(yx)| > \alpha e^{Q/pA(x)}\}| \\ &= |\{yx \in S \mid |f(yx)| > \alpha e^{Q/pA(x)}\}| e^{QA(x)} \\ &= d_f(\alpha e^{Q/pA(x)}) e^{QA(x)}. \end{aligned}$$

Therefore if $q < \infty$ then,

$$\begin{aligned} \|R_p(x)f\|_{p,q}^q &= C \int_0^\infty \alpha^{q-1} d_{R_p(x)f}(\alpha)^{q/p} d\alpha \\ &= C \int_0^\infty \alpha^{q-1} [d_f(\alpha e^{Q/pA(x)})]^{q/p} e^{QqA(x)/p} d\alpha \\ &= C \int_0^\infty (\alpha e^{Q/pA(x)})^{q-1} (d_f(\alpha e^{Q/pA(x)}))^{q/p} d(\alpha e^{Q/pA(x)}) \\ &= \|f\|_{p,q}^q \end{aligned}$$

and if $q = \infty$ then,

$$\begin{aligned} \|R_p(x)f\|_{p,\infty} &= \sup_{\alpha>0} \alpha d_{R_p(x)f}(\alpha)^{1/p} \\ &= \sup_{\alpha>0} \alpha e^{Q/pA(x)} d_f(\alpha e^{Q/pA(x)})^{1/p} \\ &= \sup_{\beta>0} \beta d_f(\beta)^{1/p} = \|f\|_{p,\infty}. \end{aligned}$$

We note that $T_\mu(f)(y) = \int_S f(yz^{-1}) e^{QA(z)} d\mu(z)$. For $f \in L^{p,q}(S)$, $g \in L^{p',q'}(S)$ (taking $q' = 1$ when $q = \infty$) we have,

$$\begin{aligned} \langle T_\mu f, g \rangle &= \int_S f * \mu(y) g(y) dy \\ &= \int_S \int_S f(yz^{-1}) e^{QA(z)} d\mu(z) g(y) dy \\ &= \int_S e^{Q/p'A(z)} \left(\int_S (R_p(z^{-1})f)(y) g(y) dy \right) d\mu(z) \\ &\leq \int_S e^{Q/p'A(z)} d\mu(z) \|f\|_{p,q} \|g\|_{p',q'}. \end{aligned}$$

We have used the fact that $A(z^{-1}) = -A(z)$ in one of the steps above. Since μ is radial and $\mathcal{R}(e^{Q/p'A(\cdot)}) = \phi_{i\gamma_p\rho}$ (see [2, 3.11]) we have (see [16, p. 70]),

$$\|T_\mu\|_{L^{p,q} \rightarrow L^{p,q}} \leq \int_S e^{Q/p'A(z)} d\mu(z) = \int_S \phi_{i\gamma_p\rho}(z) d\mu(z) = \widehat{\mu}(i\gamma_p\rho). \quad \square$$

Through similar steps one can also show that if a nonnegative radial measure μ satisfies $\int_S d\mu(x) < \infty$, then

$$\|T_\mu\|_{L^1 \rightarrow L^1} \leq \int_S d\mu(x) \quad \text{and} \quad \|T_\mu\|_{L^\infty \rightarrow L^\infty} \leq \int_S d\mu(x).$$

3.3. Spectrum of Δ . We recall that for $p \geq 1$, $\gamma_p = (2/p - 1)$ and $S_p = S_{p'} = \{z \in \mathbb{C} \mid |\Im z| \leq |\gamma_p| \rho\}$. Under the map $\Lambda : z \mapsto z^2 + \rho^2$, S_p is mapped to a parabolic region in the complex plane which we shall denote by P_p . Precisely $P_p = \Lambda(S_p)$. We note that $P_p = P_{p'}$ and if $p = 2$ then $S_p = \mathbb{R}$ and the Parabolic region reduces to the ray $[\rho^2, \infty)$. For $p \neq 2$, let S_p° and P_p° be the interiors of S_p and P_p respectively. We enlist here some information related to the spectrum of Δ which will be useful for proving our main results.

Suppose that a nonzero measurable function u satisfies $\Delta u = (\lambda^2 + \rho^2)u$ for some $\lambda \in \mathbb{C}$. We assume that for a point $x_0 \in S$, $u(x_0) \neq 0$. Let $\ell_x f$ be the left translation of a function f by $x \in S$. Since Δ commutes with the radialization operator and translations, $\mathcal{R}(\ell_{x_0} u)$ is a radial eigenfunction with the same eigenvalue $\lambda^2 + \rho^2$. Hence $\mathcal{R}(\ell_{x_0} u) = C\phi_\lambda$ (see [2, 2.5]). As $\mathcal{R}(\ell_{x_0} u)(e) = \ell_{x_0} u(e) = u(x_0)$ and $\phi_\lambda(e) = 1$ we have $\mathcal{R}(\ell_{x_0} u) = u(x_0)\phi_\lambda$. Thus from the $L^{p,q}$ -properties of ϕ_λ given above one can determine the $L^{p,q}$ -point spectrum of Δ . Precisely,

- (i) if $2 < p < \infty, 1 \leq q < \infty$, then P_p° (respectively P_p) is the $L^{p,q}$ -point spectrum (respectively the $L^{p,\infty}$ -point spectrum);
- (ii) P_1 is the L^∞ -point spectrum and P_1° is the C_0 -point spectrum;
- (iii) for $1 < p < 2, 1 \leq q \leq \infty$, the $L^{p,q}$ -point spectrum and the L^1 -point spectrum are empty;
- (iv) (ρ^2, ∞) is the $L^{2,\infty}$ -point spectrum;
- (v) for $1 \leq q < \infty$, the $L^{2,q}$ -point spectrum is empty.

It is known that for $1 \leq p \leq \infty$, the L^p -spectrum of Δ is P_p (see [2, Cor. 4.18], see also [1, 21, 25]). We restrict to the range $\{1 < p < 2\} \cup \{2 < p < \infty\}$ and take a $\lambda \in \mathbb{C} \setminus P_p$. Then we can choose p_1, p_2 satisfying $1 < p_1 < p < p_2 < 2$ or $2 < p_1 < p < p_2$ so that $\lambda \notin P_{p_1} \cup P_{p_2}$. Therefore λ is in the L^{p_1} -resolvent set as well as in the L^{p_2} -resolvent set of Δ . Using interpolation ([24, p. 197]) we conclude that λ is in the $L^{p,q}$ -resolvent set of Δ for any $1 \leq q \leq \infty$. Thus for $1 < p < \infty$ with $p \neq 2$ and $1 \leq q \leq \infty$, the $L^{p,q}$ -spectrum of Δ is a subset of P_p . In particular the $L^{p,\infty}$ -spectrum of Δ is P_p which is also the $L^{p,\infty}$ -point spectrum when $p > 2$ as mentioned above. We conclude, noting that for an operator A on a Banach space B , the spectrum of $T_t = e^{-At}$ is not necessarily in one-to-one correspondence with the spectrum of A . (See [12]).

3.4. Heat kernel and the semigroup T_t . Let h_t be the *heat kernel* which is defined as a radial function in the Harish-Chandra Schwartz space $C^p(S)$, $1 \leq p \leq 2$, by prescribing its spherical Fourier transform $\widehat{h}_t(\lambda) = e^{-t(\lambda^2 + \rho^2)}$ for all $\lambda \in \mathbb{C}$ ([2, (5.4), (5.5)]). For any $c \in \mathbb{R}$ and $T_t = e^{-t(\Delta - c)}$, $T_t f = e^{ct} e^{-t\Delta} f = e^{ct} f * h_t$ for any suitable function f . We recall (see [2, 5.50]) that the heat maximal operator on S , $Mf(x) = \sup_{t>0} |e^{-t\Delta} f(x)|$ is weak type $L^1 - L^1$ and strong type $L^\infty - L^\infty$ and hence by interpolation ([24, p. 197]), it is strong type $L^{p,q} - L^{p,q}$ for $1 < p < \infty, 1 \leq q < \infty$. From the fact that $f * h_t(x) \rightarrow f(x)$ in $C^p(S)$ and that $C^p(S)$ is dense in $L^{p,q}(S)$, the standard method of maximal function yields $f * h_t \rightarrow f$ in $L^{p,q}(S)$ as $t \rightarrow 0$. From this it is easy to see that $\|T_t f - f\|_{p,q} \rightarrow 0$ as $t \rightarrow 0$ for any $f \in L^{p,q}(S)$. That is T_t is a strongly continuous semigroup on $L^{p,q}(S)$. It is easy to verify that T_t is strongly continuous on $C_0(S)$. It is known that $f * h_t$ does not converge to f in L^∞ and the same is true for the weak- L^p spaces.

4. PROOF OF THEOREM A AND THEOREM B

proof of Theorem A. (i) It suffices to show that T_t is chaotic on $L^{p,1}(S)$, since for $1 \leq q < \infty$, $L^{p,1}(S)$ is a dense subspace of $L^{p,q}(S)$. For this proof $B = L^{p,1}(S)$.

From the description of P_p (see section 3.3) and the condition $c > c_p$, the following conclusions are immediate: $\Omega_p = (P_p^\circ - c) \cap \{z \mid \Im z > 0\}$, $\Omega_p^+ = \Omega_p \cap \{z \in \mathbb{C} \mid \Re z > 0\}$ and $\Omega_p^- = \Omega_p \cap \{z \in \mathbb{C} \mid \Re z < 0\}$ are connected non-empty open sets ; Ω_p intersects $i\mathbb{R}$ in a nondegenerate line segment. For $z \in \Omega_p$ we define the map $\Gamma(z) = \sqrt{z - \rho^2 + c}$ taking an analytic branch. Then $\Gamma(z) \in S_p^\circ$ and $\Gamma : \Omega_p \rightarrow S_p^\circ$ is holomorphic. Hence $\Gamma(\Omega_p^+)$ and $\Gamma(\Omega_p^-)$ are connected and (by the open mapping theorem) open sets in S_p .

Since for $z \in \Omega_p$, $\Gamma(z) \in S_p^\circ$, we have $\Gamma(z) = \alpha + i\gamma_r\rho$ for $\alpha \in \mathbb{R}$ and for some r satisfying $p' < r < p$. Thus every $z \in \Omega_p$ determines an unique $r = r(z) \in (p', p)$, precisely by $r = r(z) = 2\rho(\Im \Gamma(z) + \rho)^{-1}$.

We define,

$$U_1 = \{\mathfrak{P}_{\Gamma(z)}F \mid z \in \Omega_p^+, F \in L^{r'}(N), \text{ where } r = r(z)\}.$$

By (3.1.10), $U_1 \subset L^{p,1}(S)$, since $r = r(z)$. It is also clear (see section 3) that elements of U_1 satisfy $(\Delta - c)\mathfrak{P}_{\Gamma(z)}F = z\mathfrak{P}_{\Gamma(z)}F$ and hence $T_t\mathfrak{P}_{\Gamma(z)}F = e^{-tz}\mathfrak{P}_{\Gamma(z)}F$. The condition $\Re z > 0$ ensures that $T_t\psi \rightarrow 0$ as $t \rightarrow \infty$ for any $\psi \in U_1$. Thus $U_1 \subset B_0$. Since B_0 is a vector space $\text{span}(U_1) \subset B_0$.

We assume that a function $f \in L^{p',\infty}(S)$ annihilates U_1 . We fix a $z \in \Omega_p^+$ and consider the corresponding elements $\mathfrak{P}_{\Gamma(z)}F$ of U_1 . Then by our assumption $\int_S f(x)\mathfrak{P}_{\Gamma(z)}F(x)dx = 0$. This implies that $\int_N \tilde{f}(\alpha + i\gamma_r\rho, n)F(n)dn = 0$ where $\Gamma(z) = \alpha + i\gamma_r\rho$, $r = r(z)$. Noting that by (3.1.9), $n \mapsto \tilde{f}(\alpha + i\gamma_r\rho, n)$ is in $L^r(N)$, and as F is an arbitrary function in $L^{r'}(N)$ (where $r = r(z)$), we conclude that $\tilde{f}(\alpha + i\gamma_r\rho, \cdot) \equiv 0$. In this way we can show that for any $\lambda \in \Gamma(\Omega_p^+)$, $\tilde{f}(\lambda, \cdot) \equiv 0$. Recalling that for almost every fixed n , $\lambda \mapsto \tilde{f}(\lambda, n)$ is a holomorphic function on S_p° , and that $\Gamma(\Omega_p^+)$ is an open set, we conclude that $\tilde{f}(\lambda, n) = 0$ for all $\lambda \in S_p^\circ$ and for almost every $n \in N$. By (3.1.1) this implies that $f = 0$. This shows that $\text{span}(U_1)$ and hence B_0 is dense in $L^{p,1}(S)$.

Next we define:

$$U_2 = \{\mathfrak{P}_{\Gamma(z)}F \mid z \in \Omega_p^-, F \in L^{r'}(N) \text{ where } r = r(z)\} \text{ and}$$

$$U_3 = \{\mathfrak{P}_{\Gamma(z)}F \mid z \in \Omega_p \cap i\mathbb{Q}, F \in L^{r'}(N) \text{ where } r = r(z)\}.$$

Like U_1 , the sets U_2 and U_3 are also subsets of $L^{p,1}(S)$ and the elements of U_2 and U_3 satisfy $T_t\mathfrak{P}_{\Gamma(z)}F = e^{-tz}\mathfrak{P}_{\Gamma(z)}F$. Let $\{v_{z_1}, v_{z_2}, \dots, v_{z_n}\}$ be a finite subcollection of elements of U_2 with $(\Delta - c)v_{z_k} = z_kv_{z_k}$. We take a \mathbb{C} -linear combination of these elements of U_2 :

$$g = \sum_{k=1}^n a_kv_{z_k} = T_t \left(\sum_{k=1}^n a_k e^{z_k t} v_{z_k} \right).$$

Since $\Re z_k < 0$, for any $\varepsilon > 0$ we have a suitable $t > 0$, $w = \sum_{k=1}^n a_k e^{z_k t} v_{z_k}$ satisfies $\|w\|_{p,1} < \varepsilon$ and it is clear that $\|T_t w - g\|_{p,1} < \varepsilon$. This shows that $\text{span}(U_2) \subset B_\infty$. Once we note that $\Gamma(\Omega_p^-)$ is open, the denseness of U_2 follows through the same argument used for showing denseness of U_1 . Thus B_∞ is dense.

Finally it is clear that $\text{span}(U_3) \subset B_{\text{per}}$. Noting that the set $\Omega_p \cap i\mathbb{Q}$ has limit points we can apply again almost a similar argument applied for U_1 and U_2 to show that $\text{span}(U_3)$ is dense. This proves the assertion.

To prove the converse, we notice that if $c \leq c_p$ then $(P_p - c) \cap i\mathbb{R}$ has at most one point. Therefore by Proposition 2.3.2 (i), T_t is not chaotic in this case.

(ii) Since T_t is not strongly continuous on $L^\infty(S)$ and on $L^{p,\infty}(S)$, it is easy to see that T_t cannot be hypercyclic and hence not chaotic on these spaces. Since $c > c_p$, by (i) above, T_t is chaotic on $L^p(S)$ which is a subspace of $L^{p,\infty}(S)$. Therefore T_t is subspace-chaotic on $L^{p,\infty}(S)$ when $c > c_p$.

We shall show that if $c > c_\infty = 0$, then T_t is chaotic on $C_0(S)$ which is a subspace of $L^\infty(S)$. For this proof $B = C_0(S)$. We fix a $c > 0$ and define the sets, Ω_∞ , Ω_∞^+ and Ω_∞^- putting $p = \infty$ in the definition of Ω_p given in (i). Then Ω_∞ is a connected open set which intersects $i\mathbb{R}$ in a nondegenerate line segment and $\Gamma(\Omega_\infty) \subset S_1^\circ$, where the function Γ is as defined in (i). It is clear that for $z \in \Omega_\infty$, $\psi(z) = \phi_{\Gamma(z)}$ (the elementary spherical function $\phi_\lambda, \lambda = \Gamma(z)$) is an eigenfunction of $\Delta - c$ with eigenvalue z and $\psi(z) \in C_0(S)$ (see section 3).

Let $V_1 = \{\ell_y(\psi(z))(x) \mid z \in \Omega_\infty^+, y \in S\}$, where ℓ_y is the left translation; $\ell_y(\psi(z))(x) = \phi_{\sqrt{z-\rho^2+c}}(y^{-1}x)$. Then $\ell_y(\psi(z))(x)$ is also an eigenfunction of $\Delta - c$ with the same eigenvalue z and $T_t \ell_y(\psi(z))(x) = e^{-zt} \ell_y(\psi(z))(x)$. Since $\Re z > 0$, it follows that $T_t \ell_y(\psi(z))(x) \rightarrow 0$ as $t \rightarrow \infty$. Therefore $V_1 \subset B_0$. Since B_0 is a vector space we have $\text{span}(V_1) \subset B_0$. We recall that $(C_0(S))^*$ is the set of complex measures. If a nonzero measure $\mu \in (C_0(S))^*$ annihilates $\text{span}(V_1)$, then for every $z \in \Omega_\infty^+$, $\mu * \phi_{\Gamma(z)} \equiv 0$. In other words for every $\lambda \in \Gamma(\Omega_\infty^+)$, $\mu * \phi_\lambda \equiv 0$. Noting that for any fixed $y \in S$, $\lambda \mapsto \mu * \phi_\lambda(y)$ is an analytic function on S_1° and that $\Gamma(\Omega_\infty^+)$ is an open set we conclude that $\mu * \phi_\lambda(y) = 0$ for all $\lambda \in S_1$ and for all $y \in S$. This implies that $\mu = 0$. Hence $\text{span } V_1$ as well as B_0 is dense in $C_0(S)$.

We define $V_2 = \{\ell_y(\psi(z))(x) \mid z \in \Omega_\infty^-, y \in S\}$ and $V_3 = \{\ell_y(\psi(z))(x) \mid z \in \Omega_\infty \cap i\mathbb{Q}, y \in S\}$. Again the elements $\ell_y(\psi(z))(x)$ of V_2 and V_3 are eigenfunctions of $\Delta - c$ with eigenvalue z . Argument analogous to (i) now shows that $\text{span } V_2 \subset B_\infty$ and $\text{span } V_3 \subset B_{\text{Per}}$. Finally the denseness of B_∞ and B_{Per} in $C_0(S)$ will follow through the argument used above to show denseness of $\text{span } V_1$. We omit the details to avoid repetition.

Argument for the converse statement is also same as that of (i). (See Remark 2.3.3.)

(iii) First we deal with the case $2 < p < \infty$ and $1 \leq q < \infty$. To show that T_t is not hypercyclic we take a nonzero $\phi \in L^{p',q'}(S)$. We claim that $\{T_t^* \phi \mid t \geq 0\}$ is a bounded set in $L^{p',q'}(S)$. We note that $T_t^* \phi = T_t \phi = \phi * p_t$ where $p_t = e^{ct} h_t$. From Lemma 3.2.1 we see that $\|T_t^* \phi\|_{p',q'} \leq \widehat{p}_t(i\gamma_p \rho) \|\phi\|_{p',q'} = e^{(c-c_p)t} \|\phi\|_{p',q'}$. As $c \leq c_p$ the claim is established. Therefore by Proposition 2.3.2 (iii), T_t is not hypercyclic on $L^{p,q}(S)$.

For the case $2 < p < \infty$ and $q = \infty$, we take $\phi \in L^{p',1}(S)$. Then $\phi \in (L^{p,\infty})^*(S)$. By Lemma 3.2.1, $\|T_t^* \phi\|_{p',1} = \|T_t \phi\|_{p',1} \leq e^{(c-c_p)t} \|\phi\|_{p',1}$. Then

$$\|T_t \phi\|_{(L^{p,\infty}(S))^*} = \sup_{\psi \in L^{p,\infty}(S)} \frac{|\langle T_t \phi, \psi \rangle|}{\|\psi\|_{p,\infty}} \leq \frac{e^{(c-c_p)t} \|\phi\|_{p',1} \|\psi\|_{p,\infty}}{\|\psi\|_{p,\infty}} = e^{(c-c_p)t} \|\phi\|_{p',1}.$$

Rest of the argument is same as the previous case. The case of $L^\infty(S)$ can be treated analogously, taking $\phi \in L^1(S) \subset (L^\infty(S))^*$. From Remark 2.3.3 it is clear that T_t cannot be subspace-chaotic on these spaces. \square

proof of Theorem B. (i) Once we notice that for $1 \leq q \leq \infty$, the $L^{2,q}$ -spectrum of $\Delta - c$ is either empty or lies entirely on \mathbb{R} , it follows from Proposition 2.3.2 (i) that T_t is not chaotic or subspace-chaotic.

For the last part of (i), the argument is similar to what is used for the proof of Theorem A (iii). We can show that for a nonzero function $\phi \in L^{2,q'}(S)$, $\|T_t^* \phi\|_{2,q'} \leq e^{(c-\rho^2)t} \|\phi\|_{2,q'}$. Similarly for a nonzero function $\phi \in L^{2,1}(S)$, $\|T_t^* \phi\|_{(L^{2,\infty}(S))^*} \leq e^{(c-\rho^2)t} \|\phi\|_{2,1}$. Proposition 2.3.2 (iii) now proves the assertion as $c \leq \rho^2$.

(ii) To establish the non-hypercyclicity, (in view of Proposition 2.3.2 (ii)) it suffices to show that the point spectrum of $(\Delta - c)^* = (\Delta - c)$ is nonempty on the dual spaces of these spaces, which is indeed

the case as: $\phi_\lambda \in L^{2,\infty}(S)$ for any nonzero $\lambda \in \mathbb{R}$; $\phi_\lambda \in (L^{p,q}(S))^* = L^{p',q'}(S)$ for $\lambda \in S_p^\circ$ (see section 3) and $\phi_\lambda \in L^\infty(S)$ if $\lambda \in S_1$. This also shows that T_t is not subspace-chaotic on these spaces. \square

5. EXISTENCE OF PERIODIC POINTS OF T_t

As in the previous sections let $T_t = e^{-t(\Delta-c)}$ for $t \geq 0$, where $c \in \mathbb{R}$ is fixed and for any $p \geq 1$, $c_p = 4\rho^2/pp'$. It is clear that if $(\Delta - c)f = 0$ for some suitable function f , then $T_t f = f$ for all $t \geq 0$. However from the hypothesis $T_t f = f$ for some $t > 0$, it does not seem to be straightforward to conclude that $(\Delta - c)f = 0$ (see [12]). This forces us to go through a round-about argument involving Wiener Tauberian theorem (WTT) to prove Proposition 5.1.2 stated below, which is the aim of this section. We begin with the following observations.

- (a) For $2 < p \leq \infty$ and $c \geq c_p$, T_t has periodic points in $L^{p,\infty}(S)$.
- (b) For $c > \rho^2$, T_t has periodic point on $L^{2,\infty}(S)$.
- (c) For any $c \in \mathbb{R}$, T_t has no periodic point in the spaces $L^1(S)$, $L^{2,r}(S)$ with $1 \leq r \leq 2$ and $L^{p,q}(S)$ with $1 < p < 2, 1 \leq q \leq \infty$.

Indeed, the assertion (a) with $c > c_p$ is proved in Theorem A (i), (ii) since $L^{p,q}(S) \subset L^{p,\infty}(S)$ and $L^{\infty,\infty}(S) = L^\infty(S)$. For the case $c = c_p$ in (a), we recall that $\phi_{i\gamma_p\rho} \in L^{p,\infty}(S)$. It can be verified easily that for all $t \geq 0$, $T_t \phi_{i\gamma_p\rho} = \phi_{i\gamma_p\rho}$. In (b) if $\lambda \in \mathbb{R}$ satisfies $\lambda^2 + \rho^2 = c$ then $\lambda \neq 0$ and hence $\phi_\lambda \in L^{2,\infty}(S)$ (see section 3). It can be verified that $T_t \phi_\lambda = \phi_\lambda$. For (c) we first note that $L^{2,r}(S) \subset L^2$ for $1 \leq r \leq 2$. Therefore it is enough to consider the spaces $L^1(S)$, $L^2(S)$ and $L^{p,q}(S)$ with p, q in the given range. We recall that if f is in one of these spaces then its Fourier transform $\tilde{f}(\lambda, n)$ exists as a measurable function in $\lambda \in \mathbb{R}$ for almost every $n \in N$. Therefore if for some $t > 0$, $T_t f - f = 0$ where f is in any of these spaces, then taking Fourier transform we have $(e^{-t(\lambda^2 + \rho^2 - c)} - 1)\tilde{f}(\lambda, n) = 0$ for almost every $\lambda \in \mathbb{R}$ and almost every $n \in N$. From this and (3.1.1) it follows that $f = 0$.

5.1. WTT and its application. Some versions of the WTT for radial (i.e. K -biinvariant) functions in the Lorentz spaces $L^{p,q}(G/K)$, $1 < p < 2, 1 \leq q < \infty$ and in $L^1(G/K)$ were established (see [22, Theorem 6.1, Remark 6.1.1]) for the rank one symmetric spaces. It is not difficult to see that exactly the same argument (which uses the result of disk algebra following [5, Theorem 1.1]), yields the corresponding results for the radial functions on DR spaces, which we shall state below.

We recall that (see [19]) for $1 \leq p < 2$, $L^{p,1}(S) * L^{p,1}(S)^\# \subset L^{p,1}(S)$ and $\|f * g\|_{p,1} \leq C\|f\|_{p,1}\|g\|_{p,1}$ where $f \in L^{p,1}(S)$ and $g \in L^{p,1}(S)^\#$. In particular $L^{p,1}(S)^\#$ is a Banach algebra under convolution.

Proposition 5.1.1. *Let $1 < p < 2$ be fixed. Suppose that for a radial function $f \in L^{p,1}(S)$ (respectively $f \in L^p(S), 1 \leq p < 2$), its spherical Fourier transform \hat{f} satisfies,*

- (i) \hat{f} extends analytically on $S_p^\varepsilon = \{z \in \mathbb{C} \mid |\Im z| < |\gamma_p| + \varepsilon\}$ for some $\varepsilon > 0$;
- (ii) $\lim_{|\lambda| \rightarrow \infty} \hat{f}(\lambda) = 0$ on S_p^ε ;
- (iii) for all $\lambda \in S_p^\varepsilon$, $\hat{f}(\lambda) \neq 0$;
- (iv) for $t \in \mathbb{R}$, $\limsup_{|t| \rightarrow \infty} |\hat{f}(t)e^{Ke^{|t|}}| > 0$ for all $K > 0$.

Then the ideal (respectively the $L^1(S)^\#$ -module) generated by f is dense in $L^{p,1}(S)^\#$ (respectively in $L^p(S)^\#$).

Following extension is immediate: if a radial function $f \in L^{p,1}(S), 1 \leq p < 2$ satisfies conditions (i)-(iv), then the left $L^{p,1}(S)$ -module generated by f is dense in $L^{p,1}(S)$. Let us denote by M the closed left $L^{p,1}(S)$ -module generated by f in $L^{p,1}(S)$. It is clear from Proposition 5.1.1 that $M \supset L^{p,1}(S)^\#$, hence in particular $h_t \in M$ for all $t \geq 0$. We take any $g \in L^{p,1}(S)$. Then $g * h_t \in M$. Since $g * h_t \rightarrow g$

in $L^{p,1}(S)$ and M is closed, we have $g \in M$. The same argument also shows that if a radial function $f \in L^p(S)$, $1 \leq p < 2$ satisfies conditions (i)-(iv), then the left $L^1(S)$ -module generated by f in $L^p(S)$ is dense in $L^p(S)$. We shall now apply these results to prove the following.

Proposition 5.1.2. *For $p \in (2, \infty)$, $q \in [1, \infty]$ and $c < c_p$, T_t has no periodic point in $L^{p,q}(S)$. If $c < c_\infty = 0$, then T_t has no periodic point in $L^\infty(S)$.*

Proof. Since $L^{p,q}(S) \subset L^{p,\infty}(S)$ for all q , it suffices to show that the assumption $T_t f = f$ for a nonzero $f \in L^{p,\infty}(S)$ and a $t > 0$, leads to a contradiction. Indeed, this assumption implies that $e^{ct} f * h_t - f = 0$. Convolving with $h_{t'}$ for some $t' > 0$, we get $f * (e^{ct} h_{t+t'} - h_{t'}) = 0$. Let $h = (e^{ct} h_{t+t'} - h_{t'})$. Then $h \in L^{p',1}(S)$, h is radial and $f * h = 0$. For $0 < \varepsilon < c$, $\hat{h}(\lambda) = e^{-t'(\lambda^2 + \rho^2)}(e^{-t(\lambda^2 + \rho^2 - c)} - 1)$ extends analytically to S_p^ε and $\hat{h}(\lambda) \neq 0$ for all $\lambda \in S_p^\varepsilon$. Indeed, \hat{h} satisfies conditions (i)-(iv) of Proposition 5.1.1. It follows from Proposition 5.1.1 and the subsequent discussion, that the left $L^{p',1}$ -module generated by $e^{ct} h_{t+t'} - h_{t'}$ is dense in $L^{p',1}(S)$. Since for any $g_1 \in L^{p',1}(S)$, $\langle f, g_1 * h \rangle = \langle f * h, g_1 \rangle = 0$, we have for any $g \in L^{p',1}(S)$, $\langle f, g \rangle = 0$. This implies that $f = 0$, since $f \in L^{p,\infty}(S) = (L^{p',1}(S))^*$. For the case of $L^\infty(S)$, we argue the same way and use Proposition 5.1.1 for $L^1(S)$. \square

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